A Complete Time Geo/G/1 Retrial Queue Analyzation with a Second Optional Service Available

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***Abstract*:** A discrete time Geo/G/1 retrial queuing model with a single vacation and a second discretionary service was covered in this work. In this paradigm, every client receives comprehensive primary vital service from the system, and those who visit during peak hours stop being persistent. A small percentage of clients want a second optional service once first one is finished. Additionally, the system has an idle of unsystematic distance. After this vacation, the server only begins to function if a customer is present; otherwise, it remains idle until at least one customer is present. While service and vacation periods are generally dispersed, arrival times are geometrically distributed in this approach. By creating function technique we obtain PGF and also we derived an analytical expression for mean queue length.

Keywords: Arrival and duration of service, Both mandatory and voluntary services, Geo/G/1 queue for retrying, average length of the lineup, non-regular clients

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1. Introduction

Many scholars are becoming more interested in discrete queuing models as a result of the discrete queue's uses in communication systems and other related fields. Many computers and communication systems are modeled using the discrete queue approach since many events in these systems may only occur at regular intervals. Additionally, discrete queues work better for computer modeling and telecommunication systems than continuous time queues.

The use of retry queuing models is widespread in real-world scenarios. especially in computer and communication systems, telecom networks, and phone exchange. One of this queue's unique features is that customers stay in the server queue while the system serves another client, and they join an orbital retrial group where they can make many tries to receive the service. Numerous survey research publications on retrial queues have been published [1] through [4].Only continuous case retrial queues have been the subject of study in the past several years. Madan [5], Medhi [6], and Choudhury [7] have studied M/G/1 queues, where services are offered in two phases. However, Yang and Li [8] have recently expanded this research to include discrete retrial queues.

A thorough analysis of the aforementioned queue, together with an extra alternative for service devoid of system faults, has been conducted by Atencia and Moreno [9]. The topic of discrete bulk retrial queues with control of admittance was covered by Atencia, Moreno, and Artoleju [10]. Another topic covered by M. Takahashi, H. Oswa, and T. Fujisawa [11] was separate time bulk reattempt queues with non-preemptive priority. Moreno [12] examined a discrete reattempt queue with a faulty system and a long-running common system. Gautam Choudhery [13] talks about the D policy's M/G/1 backlog with two stage service. M/G/1 retrial queues with second optional services at different vacation rules and non-persistent clients were examined by Ramanath and Kalidass [14]. Wang and Zhao [15] has analyzed discrete time reattempt queue with early downs and additional option for service.

I was inspired to write this paper because we combined the single vacation with the aforementioned concepts to create a new one: a discrete reattempt queue with an extra service choice and single vacation. This concept has many applications in the real world.This article's goal is to talk about the kind of issue that comes up in customer service centers. When a consumer calls a contact center, they are initially put through to a receptionist who collects all of the customer's information and answers all of their questions (first important service). If a customer has any technical issues, they may be able to speak with a technical expert at this call center (second optional service). If not, he could be satisfied with the receptionist's response. When the contact center is busy, customers have two options: they can choose to turn off their phone for the time being or keep calling in an attempt to receive a response. The receptionist is free to take a break (vacation) or work at the contact center after answering every call. This procedure works well with the model we are looking at.

2. Model description

We examine a discrete time retrial single server queue in which the time axis is divided into intervals of equal time intervals called slots. The sequence in which the departure and arrival occur is crucial because, in a discrete queue, there is a chance that other queuing events may occur concurrently and the probability of entrance will not be zero. The literature discusses two types of policies: (i) late arrival systems (LAS) or arrival first (AF) policies are those in which an arrival happens first, and (ii) early arrival systems (EAS) or departure first (DF) policies are those in which a departure occurs first. In this paper we follow case (ii), ie we study the type only for EAS. For accurate calculation, suppose that the customer departure take place in the time interval (m^-,m) and the customer arrival and retrials take place in the time interval (m, m^+), that is the customer entry, and the retrials happen sright after instantly the opening limits and the exist immediately before the slot boundary.

With probability p customer arrival process follows Bernoulli process. At busy period the customer decided to exist the server (1-r) (probability) hence he becomes non persistent and with probability r he joins the orbit (customer in the orbit are waiting to retry services).An arriving customer turn on the service station when the system is idle. Here the busy system probability is θ and idle system probability is $\bar{\theta}$ .Once the primary service has done the customer can either receive additional optional service i.e. α probability or can be left from the server with complementary $\bar{\alpha}$ probability.

In the circle every client makes a free reattempt source and the reattempt time (the time between two progressive attempts by a similar client) obeys a mathematical distribution with probability 1 – r. For repeated customer the retrial process confirms only if, upon a specific trial, the system has no job service and the recurring client is chosen for service among all other recurring clients who are endeavoring the service around then and service station is actuated effectively.

Also, the system has break with probability η if there have been no client in the system. After returning from vacation if minimum one client available in the server the system starts its service with probability $\bar{\eta} = 1 - \eta$ otherwise the system is idle. The service times $\{t_{1,k}\}$ and $\{t_{2,k}\}$respectively of first primary and second optional services are autonomous and generally distributed with probability generating function $T_1(x) = \sum_{k=1}^{\infty} t_{1k} x^k$ and $T_2(x) = \sum_{k=1}^{\infty} t_{2k} x^k$ respectively. The nth factorial moments will be denoted by $\beta_1^{(n)}$ and $\beta_2^{(n)}$.respectively.

At last it is suppose that the vacation times, service times, retrial times and inter arrival times are mutually independent. Further, we suppose $0 \prec p \prec 1$, $0 \prec r \prec 1$,$0 \prec \theta \prec 1 and 0 \prec \eta \prec 1$in order to avoid trivial cases We use the notation to denote by $\bar{p} = 1 - p$ and $\rho = \rho_1 + \rho_2$the traffic intensity, where $\rho_1 = p\beta_{1,1}$ and $\rho_2 = \alpha p \beta_{2,1}$respectively represent the server load because of arrival in the primary and secondary optional services.

3. The Markov chain

The state of the system at time n^+ is mentioned below

$$S_n = \{X_n, \zeta_n, C_n\}$$

where X_n represents the system state ie, if the service is unoccupied then X_n =0, if the system is busy with primary service and additional service then X_n =1and X_n =2 respectively and the server is on vacation then X_n =3 and C_n , the number of clients in the retrial group. If $X_n = (1,2)$, then $\zeta_1^{(n)}$ denotes the outstanding service time of the customer which is currently in service .If $X_n = 3$ $\zeta_2^{(n)}$ represents remaining vacation time.

Subsequent to presenting the above supplementary factors of S_n the future elements of depends just on the present state. Else on the other hand, given the present express, the following state, and the advancement of the framework preceding the present state are autonomous. So it can be demonstrated that [S_n , n € C } is the Markov chain of our queuing system, where the state space is

S = $\{(0,k); k \geq 0, (j.i.k); j = 1,2,3, i \geq 1, k \geq 0\}$

4. Queue length distribution

The second optional service served to some of the customers those who are completed their first essential service exhaustively only

The limiting probabilities are defined as

$$\pi_{0,j} = \lim_{n\to\infty} \Pr\{X_n = 0, N_n = j\}$$

$$\pi_{1,k,j} = \lim_{n\to\infty} \Pr\{X_n = 1, \xi_1^{(n)} = k, N_n = j\}$$

$$\pi_{2,k,j} = \lim_{n\to\infty} \Pr\{X_n = 2, \xi_1^{(n)} = k, N_n = j\}$$

$$\pi_{3,k,j} = \lim_{n\to\infty} \Pr\{X_n = 3, \xi_2^{(n)} = k, N_n = j\}$$

The Kolmogorov equations for the stationary distribution is

$$\pi_{0,j} = \bar{p}r^k \pi_{o,j} + \bar{\alpha}\bar{p}r^k \pi_{1,1,j} + \bar{p}r^k \pi_{2,1,j} + (1-\delta_{0,k})\bar{p}\pi_{3,1,j} \quad j \geq 0,$$

$$\pi_{1,k,j} = p\theta t_{1,i}\pi_{0,j} + \bar{p}(1-r^{j+1})\theta t_{1k}\pi_{0j+1} + \bar{\alpha}p\theta t_{1k}\pi_{21j} + \bar{\alpha}\bar{p}(1-r^{j+1})\theta t_{1k}\,\pi_{1,1,j+1} + (1-\delta_{0,j})p\pi_{1k+1j-1}\,, + p\theta t_{1k}\,\pi_{2,1,j}$$

$$+\bar{p}\pi_{1,k+1,j} + \bar{p}(1-r^{j+1})\theta t_{1k}\pi_{2,1,j+1} + (1-\delta_{0,j})p\theta t_{1k}\pi_{3,1,j} + (1-r^{j+1})\overline{p\theta} t_{1k}\pi_{3,1,j+1}\,, k \geq 1, j \geq 0 \qquad (2)$$

$$\pi_{2,k,j} = (1-\delta_{0,j})p\alpha t_{2k}\pi_{1,1,j-1} + \alpha t_{2,k}\,\bar{p}\pi_{1,1,j} + (1-\delta_{0,j})p\pi_{2,k+1,j-1} + \bar{p}\pi_{2,k+1,j} + (1-\delta_{0,j})p\alpha t_{2,k}\pi_{3,1,j-1} + \bar{p}\alpha t_{2,k}\pi_{3,1j}$$

$$\pi_{3,k,j} = \bar{p}\eta v_k \pi_{1,1,j+1}\bar{\alpha}(1-r^{j+1}) + p\eta v_k \pi_{1,1,j}\;\bar{\alpha} + \bar{p}\eta v_k \pi_{2,1,j+1}\bar{\alpha}(1-r^{j+1}) + p\eta v_k \pi_{2,1,j} + \bar{p}\pi_{3,k+1,j} + p\pi_{3,k+1,j-1}(1-\delta_{0,j})$$

$k \geq 1, j \geq 0$ (3)

$$+\bar{p}\eta v_k \pi_{j+1}(1-r^{j+1}) + p\eta v_k \pi_{0,j} + \bar{p}\eta v_k \pi_{3,1,j+1}(1-r^{j+11}) + p\eta v_k \pi_{3,1,j} \qquad k \geq 0, j \geq 0 \qquad (4)$$

The normalization condition is'

$$\sum_{j=0}^{\infty} \pi_{0,j} + \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \pi_{1,k,j} + \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \pi_{2,k,j} + \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \pi_{3,k,j} = 1$$

We introduce following generating function to solve the above system of equations

$$\Phi_0(z) = \sum_{j=0}^{\infty} \pi_{0,j} z^j \quad \Phi_1(x,z) = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \pi_{1,k,j} x^k z^j$$

$$\Phi_2(x,z) = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \pi_{2,k,j} x^k z^j \quad \Phi_3(x,z) = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \pi_{3,k,j} x^k z^j$$

and also we define auxiliary generating function as follows

$$\Phi_{1,k}(z) = \sum_{j=0}^{\infty} \pi_{1,k,j} z^j \quad \Phi_{2,k}(z) = \sum_{j=0}^{\infty} \pi_{2,k,j} z^j \quad \Phi_{3,k}(z) = \sum_{j=0}^{\infty} \pi_{3,k,j} z^j$$

Multiply both sides by z^j and taking summation over j in equation (1), we get

$$\Phi_0(z) = \bar{p}\Phi_0(rz) + \bar{p}\bar{\alpha}\Phi_{1,1}(rz) + \bar{p}\Phi_{2,1}(rz) + \bar{p}\Phi_{3,1}(rz) \qquad (5)$$

Multiply both sides by z^j and taking summation over j in equation (2), we get

$$\Phi_{1,k}(z) = (\tau)\Phi_{1,k+1}(z) + \frac{(\tau)}{z}\theta t_{1,k}\left[\Phi_0(z) + \bar{\alpha}\Phi_{1,1}(z) + \Phi_{2,1}(z) + \Phi_{31}(z)\right] - \frac{\bar{p}}{z}\theta t_{1,k}\left[\Phi_0(rz) + \bar{\alpha}\Phi_{1,1}(rz) + \Phi_{2,1}(rz)\right]$$

$$-\frac{\bar{p}}{z}\theta t_{1,k}\left[\Phi_0(rz)+\bar{\alpha}\Phi_{1,1}(rz)+\Phi_{2,1}(rz)+\Phi_{3,1}(rz)\right] \text{where } \tau = \bar{p} + pz \quad (6)$$

Multiply both sides by z^j and taking summation over j in equation (3), we get

$$\Phi_{2,k}(z)=(\tau)\Phi_{2,k+1}(z)+(\tau)\alpha t_{2,k}\Phi_{1,1}(z)+(\tau)\alpha t_{2,k}\Phi_{3,1}(z) \quad (7)$$

Multiply both sides by z^j and taking summation over j in equation (4), we get

$$\Phi_{3,k}(z)=\frac{\eta v_k(\tau)}{z}\left[\phi_0(z)+\bar{\alpha}\phi_{11}(z)+\varphi_{21}(z)+\varphi_{31}(z)\right]+(\tau)\varphi_{3k+1}(z)-\frac{p\eta v_k}{z}\varphi_0(z) \quad (8)$$

Using (5) in (6) and after some algebraic simplification, we obtain

$$\Phi_{1,k}(z)=(\tau)\Phi_{1,k+1}(z)-\frac{1-z}{z}\theta t_{1,k}\Phi_0(z)p+\frac{\tau}{z}\theta t_{1,k}\left[\bar{\alpha}\Phi_{1,1}(z)+\Phi_{21}(z)+\Phi_{3,1}(z)\right] \quad (9)$$

Using (5) in (8) and doing some simplification, we obtain

$$\Phi_{3,k}(z)=(\tau)\Phi_{3,k+1}(z)-\frac{1-z}{z}\eta v_k\Phi_0(z)+\frac{\tau}{z}\eta v_k\left[\bar{\alpha}\Phi_{11}(z)+\Phi_{21}(z)+\Phi_{31}(z)\right] (10)$$

Multiply both sides by x^k and taking summation over k and doing some simplification in equation (9), we obtain

$$\Phi_1(x,z)\left[\frac{x-\tau}{x}\right]=\frac{\tau}{z}\left[\bar{\alpha}\theta T_1(x)-z\right]\Phi_{11}(z)-\frac{1-z}{z}p\Phi_0(z)T_1(x)+\frac{(\tau)}{z}\theta T_1(x)\left[\varphi_{21}(z)+\varphi_{31}(z)\right] \quad (11)$$

Multiply both sides by x^k and taking summation over k and doing some simplification in equation (7), we obtain

$$\Phi_2(x,z)\left(\frac{x-\tau}{x}\right)=(\tau)\alpha\Phi_{1,1}(z)T_2(x)-(\tau)\Phi_{2,1}(z)+\alpha(\tau)T_2(x)\varphi_{31}(z) (12)$$

Multiply both sides by x^k and taking summation over k and doing some simplification in equation (10), we obtain

$$\Phi_3(x,z)\left(\frac{x-\tau}{x}\right)=\frac{\tau}{z}\Phi_{3,1}(z)[v(x)-z]-\frac{1-z}{z}\eta\varphi_0(z)v(x)+\frac{\tau}{z}\eta v(x)\left[\bar{\alpha}\varphi_{11}(z)+\varphi_{21}(z)\right] \quad (13)$$

In equation (11) put $x = \tau$ and solving for $\Phi_{1,1}(z)$, we get

$$\Phi_{1,1}(z)=\frac{T_1(x)\theta\left[(1-z)p\Phi_0(z)-\tau(\Phi_{2,1}(z)+\Phi_{3,1}(z)\right]}{\tau\left[\bar{\alpha}\theta T_1(\bar{p}+pz)-z\right]} \quad (14)$$

In equation (12) put $x = \bar{p} + pz$ and solving for $\Phi_{2,1}(z)$, we get

$$\Phi_{2,1}(z)=\frac{\tau\alpha(\Phi_{1,1}(z)+\Phi_{3,1}(z))T_2(x)}{\tau} \quad (15)$$

In equation (13) put $x = \bar{p} + pz$ and solving for $\Phi_{3,1}(z)$, we get

$$\Phi_{3,1}(z)=\frac{\eta v(x)\left[(1-z)\phi_0(z)-\tau(\bar{\alpha}\phi_{11}(z)+\phi_{31}(z))\right]}{[v(x)-z]\tau}$$

(16) Using (16) in (15) and solving for $\Phi_{2,1}(z)$

$$\Phi_{2,1}(z)=\frac{\tau\alpha T_2(x)\varphi_{11}(z)(v(x)-z)+\alpha T_2(x)v(x)\varphi_0(z)\left[(1-z)-\bar{\alpha}\tau\right]}{\tau\{[v(x)-z]+\alpha T_2(x)\eta v(x)\}} \quad (17)$$

Using (17) in (16) and solving for $\Phi_{3,1}(z)$

$$\Phi_{3,1}(z)=\frac{\eta v(x)\left[\varphi_{11}(z)\right]\left[(1-z)-\overline{\alpha}\tau\right]\{[[v(x)-z]+\alpha T_2(x)]\eta v(x)]-\alpha T_2(x)v(x)\}}{[[v(x)-z]+\alpha T_2(x)\eta v(x)][v(x)-z]\tau} \tag{18}$$

Using (17) in (18) in (14) and solving for $\Phi_{1,1}(z)$, we get

$$\Phi_{1,1}(z)=\frac{\begin{array}{c}T_1(x)p\theta(1-z)[[v(x)-z]+\alpha T_2(x)]+v(x)\left[(1-z)-\overline{\alpha}\tau\right](1-\alpha T_2(x))\\ \{[[v(x)-z]+\alpha T_2(x)\eta v(x)]-\alpha z T_2(x)\}\tau\left(\overline{\alpha}\theta T_1(x)-z\right)\end{array}}{\tau\theta T_1(x)\left[\overline{\alpha}+\alpha T_2(x)-z\right]}\Phi_0(z) \tag{19}$$

Using (19) in (17) and solving for $\Phi_{2,1}(z)$, we get

$$\Phi_{2,1}(z)=\frac{\begin{array}{c}\alpha T_2(x)\varphi_0(z)[T_1(x)p\theta(1-z)[(v(x)-z)+\alpha T_2(x)\eta v(x)]]+\left(\overline{\alpha}\theta T_1(x)-z\right)(1-\alpha T_2(x))\}\\ \{v(x)\left[(1-z)-\overline{\alpha}\tau\right]\{[(v(x)-z)+\alpha T_2(x)\eta v(x)]\tau+1\}\end{array}}{[(v(x)-z)+\alpha T_2(x)\eta v(x)]\tau\left(\overline{\alpha}\theta T_1(x)-z\right)(1-\alpha T_2(x))} \tag{20}$$

Using (19) in (18) and solving for $\Phi_{3,1}(z)$, we get

$$\Phi_{3,1}(z)=\frac{\left[(1-z)-\overline{\alpha}\tau\right]\eta v(x)\varphi_0(z)}{[v(x)-z]}\left[\frac{1+\alpha T_2(x)v(x)\tau}{\tau}\right]$$

$$+\frac{\alpha T_2(x)\eta v(x)T_1(x)p\theta(1-z)\{[v(x)-z]+\alpha T_2(x)\}\varphi_0(z)}{\{[v(x)-z]+\alpha T_2(xz)\eta v(x)\}\tau\left[\overline{\alpha}\varphi T_1(x)-z\right][v(x)-z][1-\alpha T_2(x)]}$$

$$-\frac{\alpha T_2(x)v(x)\left[(1-z)-\overline{\alpha}(\tau)\right]\varphi_0(z)}{\{[v(x)-z]+\alpha T_2(x)\eta v(x)\}[v(x)-z]}\left[\frac{v(x)}{\tau}+\alpha T_2(x)\eta v(x)\right]$$

Now, $\varphi_0[z]=\overline{p}\varphi_0[rz]+\overline{p}\varphi_{11}[rz]\overline{\alpha}+\overline{p}\varphi_{21}[rz]+\overline{p}\varphi_{31}[rz]$

$\varphi_0(z)=\overline{p}\Phi_0(rz)G(rz)$

Where G(rz)= [1+A+B+C]

The expression A,B and C are given below

$$A=\frac{\begin{array}{c}T_1\left(\overline{p}+prz\right)p\theta(1-rz)\{v\left(\overline{p}+prz\right)-rz\}[v\left(\overline{p}+prz\right)-rz]+\alpha T_2\left(\overline{p}+prz\right)v\left(\overline{p}+prz\right)\left(\overline{\alpha}+\eta\right)]\}\\ +T_1\left(\overline{p}+prz\right)p\theta(1-rz)\{\alpha T_2\left(\overline{p}+prz\right)[2[v\left(\overline{p}+prz\right)-rz]+\alpha T_2\left(\overline{p}+prz\right)v\left(\overline{p}+prz\right)(2+\eta)]\}\end{array}}{\left(\overline{p}+prz\right)\left(\overline{\alpha}\theta T_1\left(\overline{p}+prz\right)-rz\right)\left(1-\alpha T_2\left(\overline{p}+prz\right)\right)}$$

$$B=\frac{v\left(\overline{p}+prz\right)\left[(1-rz)-\overline{\alpha}\left(\overline{p}+prz\right)\right]}{\left(\overline{p}+prz\right)\left[v\left(\overline{p}+prz\right)-rz\right]\left\{\left[v\left(\overline{p}+prz\right)-rz\right]+\alpha T_2\left(\overline{p}+prz\right)\eta v\left(\overline{p}+prz\right)\right\}}$$

In (12) substitute the values of $\varphi_{11}(z)$, $\varphi_{21}(z)$, $\varphi_{31}(z)$, $\varphi_0(z)$ and on simplification we get

$$\Phi_1(x,z)=\frac{x\left(\bar{\alpha}\theta T_1(x)-z\right)\varphi_0(z)}{z^2(x-\tau)}\left\{\frac{\begin{array}{l}T_1(x)p\theta(1-z)[[v(x)-z]+\alpha T_2(x)]\\+\tau v\left(\bar{p}+pz\right)\{[v(x)-z]+\alpha T_2(x)\eta v(x)\}\left(\bar{\alpha}\theta T_1(x)-z\right)[1-\alpha T_2(x)]\end{array}}{\{[v(x)-z]+\alpha T_2(x)\eta v(x)\}\left(\bar{\alpha}\theta T_1(x)-z\right)[1-\alpha T_2(x)]}\right\}$$

Use the values of $\varphi_{11}(z)$, $\varphi_{21}(z)$, $\varphi_{31}(z)$, $\varphi_0(z)$ in equation (11) and on simplification we obtain

$$\phi_2(x,z)=\frac{x}{(x-(\tau))}\varphi_0(z)\left[\frac{\alpha T_2(x)s_1(x)p\theta(1-z)\{[v(x)-z]+\alpha T_2(x)\}\{[v(x)-z]+2\alpha T_2(x)\eta v(x)\}}{\left[\bar{\alpha}\theta T_1(x)-z\right]\{[v(x)-z]+\alpha T_2(x)\eta v(x)\}[v(x)-z][1-\alpha T_2(x)]}\right]$$

$$+\frac{x}{(x-(\tau))}\varphi_0(z)\left\{\frac{\alpha T_2(x)\left[(1-z)-\bar{\alpha}(\tau)\right]+\{[v(x)-z]+2\alpha T_2(x)\eta v(x)\}(\tau)-\alpha T_2(x)v(x)[1+\alpha T_2(x)v(x)]}{\{[v(x)-z]+\alpha T_2(x)\eta v(x)\}[v(x)-z]}\right\}$$

Use the values of $\varphi_{11}(z)$, $\varphi_{21}(z)$, $\varphi_{31}(z)$, $\varphi_0(z)$ in equation (13) and on simplification we obtain

$$\phi_3(x,z)=\frac{x}{z(x-(\tau))}\varphi_0[z]\left\{\frac{\begin{array}{l}p\theta(1-z)T_1(x)\{[v(x)-z]+\alpha T_2(x)\}\{\alpha T_2(x)\eta v(x)[v(x)-z]\}\\+p\theta(1-z)T_1(x)\{[v(x)-z]+\alpha T_2(x)\}\{[v(x)-z]+\alpha T_2(x)\eta v(x)\}\eta v(z)\bar{\alpha}\\+\{[v(x)-z]+\alpha T_2(x)\eta v(x)\}\eta v(z)\alpha T_2(x)\end{array}}{\{[v(x)-z]+\alpha T_2(x)\eta v(x)\}[v(x)-z](\tau)}\right\}$$

$$+\frac{x}{z(x-(\tau))}\varphi_0[z]\left\{\frac{\left[(1-z)-\bar{\alpha}(\tau)\right]v(x)\left\{\begin{array}{l}[1+\alpha T_2(x)\eta v(x)(\tau)[v(x)-z]\eta v(z)]\\+\bar{\alpha}\{[v(x)-z]+\alpha T_2(x)\eta v(x)\}+\alpha s_2(x)\left(\bar{\alpha}+1\right)\end{array}\right\}}{[v(x)-z](\tau)}\right\}$$

$$+\frac{x}{z(x-(\tau))}\varphi_0[z]\left\{\frac{\begin{array}{l}\alpha s_2(x)\{\left[(1-z)-\bar{\alpha}(\varsigma)\right]v(x)-(\tau)^2\alpha T_2(x)\}\eta v(z)[v(x)-z]\\-\left[(1-z)-\bar{\alpha}(\tau)\right]v(x)[v(x)-z][v(x)+\alpha T_2(x)\eta v(x)(\tau)]\end{array}}{[v(x)-z](\tau)\{[v(x)-z]+\alpha T_2(x)\eta v(x)\}}\right\}$$

We obtain the probability generating function $\Phi(z)$ by

$$\Phi(z)=\phi_0(z)+\phi_1(1,z)+\phi_2(1,z)+\phi_3(1,z)$$

$$\varphi(z)=\frac{\begin{array}{l}p\theta(1-z)T_1(x)[[v(x)-z]+\alpha T_2(x)][[v(x)-z]+\alpha T_2(x)\eta v(x)]\\ \left\{\frac{1}{z}\left[\bar{\alpha}\theta-z+\alpha T_2(x)\right]+\alpha sT(x)\eta v(x)[1+(v(x)-z)]+\eta v(z)\left[\bar{\alpha}+\alpha T_2(x)\right]\right\}\end{array}}{z[1-(\tau)]\left[\bar{\alpha}\theta T_1(x)-z\right][1-\alpha T_2(x)][[v(x)-z]+\alpha T_2(x)\eta v(x)]}\varphi_0[z]$$

$$+\frac{\varphi_0[z]\left\{\frac{\bar{\alpha}\theta-z}{z}[[v(x)-z]+\alpha T_2(x)\eta v(x)](\tau)v(x)[1-\alpha T_2(x)]\left[\bar{\alpha}\theta T_1(x)-z\right]\right\}}{z[1-(\tau)]\left[\bar{\alpha}\theta T_1(x)-z\right][1-\alpha T_2(x)]}$$

$$+\frac{\varphi_0[z]\{v(x)[v(x)-z][\alpha T_2(x)(\tau)[1+\alpha T_2(x)\eta v(x)(\tau)]+[1+\alpha T_2(x)\eta v(x)(\tau)][[v(x)-z]+\alpha T_2(x)\eta v(x)]]\}}{z[1-(\tau)][[v(x)-z]+\alpha T_2(x)\eta v(x)][v(x)-z]}$$

$$-\frac{\varphi_0[z]\alpha s_2(x)\left\{\begin{array}{l}s_1(x)(\tau)^2[v(x)-z]+(\tau)\left[(1-z)+\bar{\alpha}(\tau)\right][v(x)+\alpha T_2(x)\eta v(x)(\tau)5]\\ -z\left\{\begin{array}{l}T_1(x)p\theta(1-z)[[v(x)-z]+\alpha T_2(x)\eta v(x)]^2+v(x)\left[(1-z)+\bar{\alpha}(\tau)\right]\\ {[[v(x)-z]+\alpha T_2(x)\eta v(x)]^2(\tau)\left[\bar{\alpha}\theta T_1(x)-z\right][1-\alpha T_2(x)]}\\ +z(\tau)[[v(x)-z]+\alpha T_2(x)\eta v(x)]\left[\bar{\alpha}\theta T_1(x)-z\right][1-\alpha T_2(x)]\end{array}\right\}[v(x)-z]\\ -\left\{\begin{array}{l}[v(x)\left[(1-z)+\bar{\alpha}(\tau)\right]-(\tau)^2\alpha T_2(x)]\eta v(z)[v(x)-z]\\ -\left[v(x)\left[(1-z)+\bar{\alpha}(\tau)\right][v(x)-z]\right][v(x)+\alpha T_2(x)\eta v(x)(\tau)]\end{array}\right\}\end{array}\right\}}{z[1-(\tau)][[v(x)-z]+\alpha T_2(x)\eta v(x)][v(x)-z]}$$

5. Steady state condition

For the system under discussion the steady state is obtained from $\Phi(1)=1$, and is obviously less than one, which is given by

$$\Phi(1)=\phi_0(1)+\phi_1(1,1)+\phi_2(1,1)+\phi_3(1,1)=1$$

$$\varphi(1)=\frac{2\bar{\alpha}p\eta(\bar{\alpha}\theta-1)[pE(v)-1]}{\begin{array}{l}\phi_0(1)(\bar{\alpha}\theta-1)\left[2\bar{\alpha}p\eta[pE(v)-1]+2\eta(D)+2\bar{\alpha}[pE(v)-1](E)+2\bar{\alpha}\eta(F)+2\bar{\alpha}(G)\right]\\ +\phi_0{}'(1)[(H)]+\phi_0{}''(1)[(I)]\end{array}}$$

Where D is given by

$$G=[E(v)p-1]\left\{\alpha p[3E(T_2)+2]+1+\alpha p\theta+\alpha\bar{\alpha}(\bar{\alpha}\theta-1)+2p\eta[E(v)+E(T_2)]+\alpha\right\}+$$

$$E(T_2)\left\{\begin{array}{l}\bar{\alpha}p^2[1+2\alpha\eta]+(1+\bar{\alpha}p)(1+2\alpha p)+2p^2(\alpha\bar{\alpha}+\alpha^2\eta\theta)+\alpha p(1-\alpha)(\bar{\alpha}\theta-1)[2p+\alpha\eta+1]\\ -\alpha^2p(\bar{\alpha}\theta-1)[\alpha\eta p+2]-\alpha^2p\eta+E(s_2)\left[\alpha^2p\eta[1-(\bar{\alpha}\theta-1)]-\alpha p(\bar{\alpha}\theta-1)\right]+E(s_1)\alpha\bar{\alpha}p\eta\end{array}\right\}+$$

$$E(v)\left\{\bar{\alpha}p^2[1+2\alpha\eta]+p(1+\bar{\alpha}p)[1+2\alpha\eta]+\alpha^2p^2\eta\theta+\alpha\bar{\alpha}(\bar{\alpha}\theta-1)+E(T_2)\left[\alpha p^2\right][1+2\alpha\eta]+\alpha^2p^2\eta(2-\alpha)\right\}+$$

$$E(T_1)\alpha^2p\eta\theta+\left[\bar{\alpha}\theta E(T_1)p-1\right]\left[\alpha\bar{\alpha}\left[\alpha\eta+2p^2E(s_2)+1+\alpha\eta p\right]\right]+p(1+\bar{\alpha}p)[1+2\alpha\eta]+$$

$$2\left\{\begin{array}{l}E(T_2(T_2-1))p^2\left[\begin{array}{l}\alpha\bar{\alpha}(1+2\alpha\eta)+(\bar{\alpha}\theta-1)\left[(1-\alpha)\left[\alpha\eta+(\bar{\alpha}\theta-1)\right]+\alpha\eta\left[\eta(1-2\alpha)-\alpha^2\right]\right]+(1+\alpha)+2\alpha p^2-\alpha\\ \left[\bar{\alpha}\theta E(s_1)p-1\right]\end{array}\right]+\\ E(v(v-1))p^2\left[(\bar{\alpha}\theta-1)(1-\alpha)(1+2\alpha\eta)+(1+2\alpha)\right]+E(T_1(T_1-1))p^2\left[\alpha^2(1-\alpha)^2\theta(1+\eta)\right]\end{array}\right\}$$

$-$

$$H = 2\left\{\begin{array}{l} [E(v)p-1]\left[2\alpha+\eta-\alpha\bar{\alpha}(\bar{\alpha}\theta-1)^2\eta+\bar{\alpha}^3(\bar{\alpha}\theta-1)(1+\alpha\eta)(1+\eta)+2\alpha\bar{\alpha}(\bar{\alpha}\theta-1)+\alpha\bar{\alpha}(\bar{\alpha}\theta-1)\eta p+\eta\right] \\ +E(T_2)P\left(\begin{array}{l}\bar{\alpha}+2\bar{\alpha}\eta\alpha+\alpha\bar{\alpha}(\bar{\alpha}\theta-1)(1+\eta)--\alpha^3\eta(\bar{\alpha}\theta-1)-(\bar{\alpha}\theta-1)(\overline{\alpha^2}+\alpha^2)-2\alpha+2\overline{\alpha^2}\alpha\eta p(\bar{\alpha}\theta-1) \\ -(1+\alpha)+\alpha\overline{\alpha^2}(\bar{\alpha}\theta-1)^2-\bar{\alpha}(\bar{\alpha}\theta-1)^2\alpha^2(1+\alpha\eta)+\alpha\bar{\alpha}^2(\bar{\alpha}\theta-1)\end{array}\right) \\ E(v)p\left[\begin{array}{l}\bar{\alpha}p(1+\alpha\eta)+\alpha^2p(1+\bar{\alpha}\alpha(\bar{\alpha}\theta-1))+\bar{\alpha}\alpha\eta p-1-\alpha p+2\overline{\alpha^2}(\bar{\alpha}\theta-1)+\alpha^2\bar{\alpha}^2(\bar{\alpha}\theta-1)(\alpha+(\bar{\alpha}\theta-1)) \\ +(\bar{\alpha}\theta-1)\alpha^2+\bar{\alpha}^2\alpha\eta(\bar{\alpha}\theta-1)+2\alpha\bar{\alpha}(\bar{\alpha}\theta-1)\end{array}\right] \\ +\left[\bar{\alpha}\theta E(T_1)p-1\right]\left[\alpha^2\bar{\alpha}\eta\left[1+\bar{\alpha}(\bar{\alpha}\theta-1)\right]\right]+\{\bar{\alpha}p+(1+\alpha)(1+\bar{\alpha}p)+\alpha p\eta(2\bar{\alpha}+\alpha\theta)+ \\ +(\bar{\alpha}\theta-1)\bar{\alpha}(\alpha\eta p+\alpha+p)\}(\alpha+\bar{\alpha}\theta)+\alpha^2\bar{\alpha}(\bar{\alpha}\theta+1) \end{array}\right\}$$

amd $I=\{\bar{\alpha}(1+\alpha\eta)[1+\alpha(\bar{\alpha}\theta-1)]-(1+\alpha)\}$

6. performance measure

Mean queue length is obtained below. From the PGF we get mean queue length by differentiating this PGF with respect to z and then put z = 1

$$E(q)=\frac{2\bar{\alpha}p\eta(\bar{\alpha}\theta-1)[pE(v)-1]}{\phi_0(1)(\bar{\alpha}\theta-1)\left[2\bar{\alpha}p\eta[pE(v)-1]+2\alpha\eta(D)+2\bar{\alpha}[pE(v)-1](E)+2\bar{\alpha}\eta(F)+2\bar{\alpha}(G)\right]} \\ +\phi_0{}'(1)[(H)]+\phi_0{}''(1)[(I)]$$

Where the terms D, E, F, G, H and I are mentioned in the earlier section steady state condition.

7. Particular case

When the vacation is zero, the PGF of first primary and additional option that is second optional our model are reduced respectively into

$$\phi_1(x,z)=\frac{T_1(x)-T_1(\bar{p}+pz)}{x-(\bar{p}+pz)}\frac{p\theta x(1-z)}{\theta T_1(\bar{p}+pz)\left[\bar{\alpha}+\alpha T_1(\bar{p}+pz)\right]-z}\phi_0(z)$$

$$\phi_2(x,z)=\frac{T_2(x)-T_2(\bar{p}+pz)}{x-(\bar{p}+pz)}\frac{p\theta x(1-z)T_2(\bar{p}+pz)}{\theta T_2(\bar{p}+pz)\left[\bar{\alpha}+\alpha T_2(\bar{p}+pz)\right]-z}\phi_0(z)$$

and PGF of the orbit of our model is reduced into

$$\Phi(z)=\phi_0(z)+\phi_1(1,z)+\phi_2(1,z)+\phi_3(1,z)$$

$$\Phi(z)=\frac{\theta(1-z)T_1(\bar{p}+pz)(\bar{\alpha}+\alpha T_2(\bar{p}+pz))}{\theta T_1(\bar{p}+pz)\left[\bar{\alpha}+\alpha T(\bar{p}+pz)\right]-z}\phi_0(z)$$

Which is PGF of discrete time Geo/G/1 retrial queue with second optional service Wang and Zhao (15)

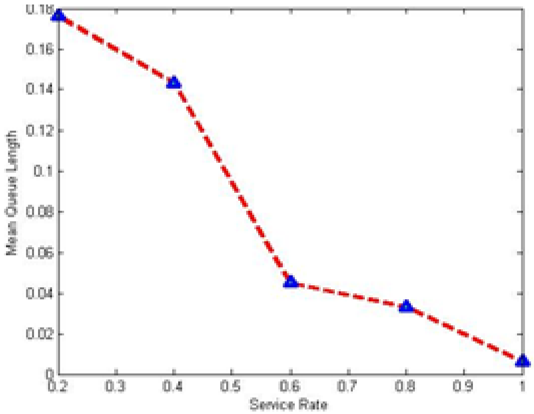
8. Numerical examples

This section has numerical examples are briefly discussed in two different cases. In both of these two cases mean queue length is investigated in following manner

1. The result on mean queue length when the arrival rate is increases
2. The result on mean queue length when the service rate is increases

Case (I): In case (I), customer arrival times ,. Service times, vacation times are all geometrically distributed.

1. When the customer arrival rate rises the result on mean queue length is investigated below with the following index with graph.



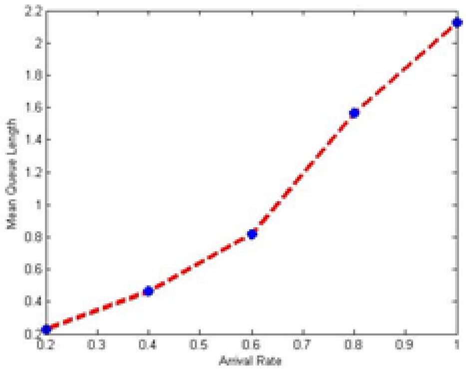
Graph 1.1 Arrival rate Vs Mean queue length

We notice that mean queue length increases when arrival rate rises which is inferred by above index and graph

2.When the server time rises the result on mean queue length is investigated below with the following index with graph.

Table 1.2

Arrival rate	$\varphi(1)$	E(Q)
0	.31	0.087
.2	.47	0 .293
.4	.609	0 .795
.6	.74	1.49
.8	.87	2.27



Graph 1.2 Service rate Vs Mean queue length

Table 2.1

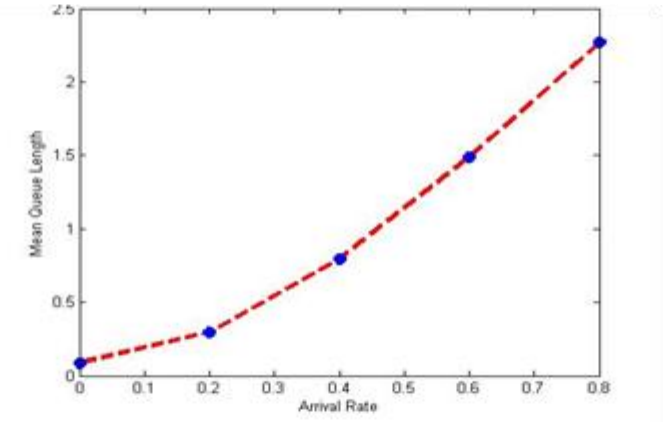
Service rate	$\varphi(1)$	Mean queue length
0.2	.71	.176
0.4	.59	.143

0.6	.41	.045
0.8	.34	.033
1.0	.21	.006

As the service rate increases, we observe that the mean wait length decreases, as indicated by the above index and table.

Case (II): In this scenario, the arrival times of customers follow a geometric distribution. The Poisson distribution is followed by service and vacation times.

1The impact of an increase in customer arrival rate on mean wait length is examined in the accompanying index and graph.



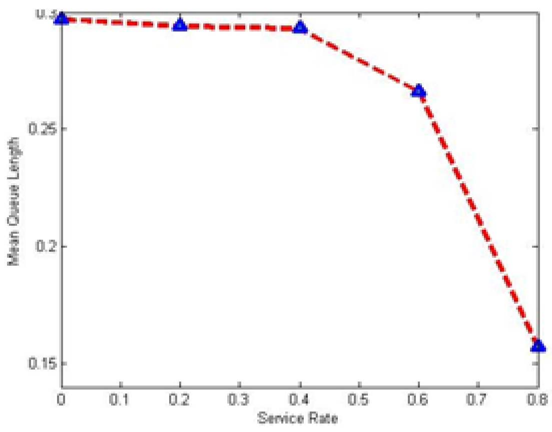
Graph 2.1 Arrival rate Vs Mean queue length

We notice that mean queue length increases when arrival rate rises which is inferred by above index and graph

2.When the server time rises the result on mean queue length is investigated below with the following index with graph.

Table 2.2

Service rate	$\varphi(1)$	**E(Q)**
0	.93	0.297
.2	.74	0.294
.4	.611	0.293
.6	.47	0.266
.8	.29	0.157



Graph 2.2 Service rate Vs Mean queue length

we notice that mean queue length decrease when service rate rises which is inferred by above index and table.

Conclusion

Here, we've examined the characteristics of a discrete time, single-vacation Geo/G/1 retrial queue with a non-persistent client and an extra service option. The PGF has been developed in this model utilizing the producing function approach. An analytical equation for mean queue length has been obtained in performance measure. Expected wait length outcomes are examined in numerical scenarios with different arrival and service rates. The majority of real-world scenarios may be applied to this paradigm.

References.

[1] J.R.Artalejo, Accessible bibliography on retrial queues, Mathematical and computer modelling, 30 (1996), 1-6
[2] J.R.Artalejo, A classified bibliography of research on retrial queues progress in 1990 – 1999, Top 7 (1999), 187-211
[3] J.R.Artalejo and Gomez – Corral, Retrial queuing system, Springer, 2008
[4] C.I.Falin, Templet on retrial queues, Chapman and Hall, London, 1997
[5] K.C.Madan, An M/G/1 queue with second optional service, Queueing system, 34 (2000), 37-46
[6] J.Medhi, A single server poission input queue with a second optional channel. Queueing systems, 42 (2002), 239 – 242
[7] G.Choudhury, Some aspects of an M/G/1 queueing system with optional second service. Top, 11 (2003), 141 – 150
[8] T.Yang, H.Li, On the steady state queue size distribution of the discrete time Geo/G/1 queue with repeated customers, Queueing system, 21 (1995) 199-215
[9] I.Atencia, P.Moreno, A discrete time Geo/G/1 retrial queue with general retrial times, Queueing system, 48 (2004), 5-21
[10] Gautam Choudhury, An M/G/1 queue with two phase service under D – policy, Information and Management sciences vol -16, No 4 (2005) 1-15
[11] Kasthuri Ramanath and K.Kalidass, An M/G/1 retrial queue with non persistent customers, a second optional service and different vacation policies, Applied Mathematical sciences, vol 4, No 40, (2010), 1967 – 1974.
[12] Jinting Wang, Qing Zhao, A discrete time Geo/G/1 queue with starting failures and second optional service, Computer and Mathematics with application 53 (2007) 115 – 127.